

DISCRETE COMPONENTS IN RESTRICTION OF UNITARY REPRESENTATIONS OF RANK ONE SEMISIMPLE LIE GROUPS

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ABSTRACT. We consider spherical principal series representations of the semisimple Lie group of rank one $G = SO(n, 1; \mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. There is a family of unitarizable representations π_ν of G for ν in an interval on \mathbb{R}^+ , the so-called complementary series, and subquotients or subrepresentations of G for ν being negative integers. We consider the restriction of (π_ν, G) under the subgroup $H = SO(n - 1, 1; \mathbb{K})$. We prove the appearing of discrete components. The corresponding results for the exceptional Lie group $F_{4(-20)}$ and its subgroup $Spin(8, 1)$ are also obtained.

1. INTRODUCTION

The study of direct components in the restriction to a subgroup $H \subset G$ of a representation (π, G) is one of major subjects in representation theory. Among representations of a semisimple Lie groups G there are two somewhat opposite classes, the discrete series and the complementary series; the former appear in the decomposition of $L^2(G)$ and can be treated algebraically, whereas the latter do not contribute to the decomposition and their study involves more analytic issues. The study of restriction of discrete series representations has been studied intensively; see e.g. [19, 15] and references therein. Motivated by some related questions of [1, 2] Speh and Venkataramana [23] studied the restriction of a complementary series representation of $SO(n, 1)$ under the subgroup $SO(n - 1, 1)$. It is approved there, for relatively small parameter ν (in our parametrization), the complementary series π_ν of $SO(n - 1, 1)$ appears discretely in the complementary series π_ν of $SO(n, 1)$ with the same parameter ν . They construct the imbedding of the complementary series of $SO(n - 1, 1)$ into π_ν of $SO(n, 1)$ by using non-compact realizations of the representations as spaces of distributions on Euclidean spaces and by extending distributions on \mathbb{R}^{n-2} to \mathbb{R}^{n-1} . Similar results are also obtained for complementary series of differential forms. For $n = 3$ the same result is proved [23] by using the compact picture and $SU(2)$ -computations; see also [18] where a full decomposition is found.

In the present paper we shall study the branching of complementary series of G for all rank one Lie groups G under a symmetric subgroup H . More precisely we prove the appearance of discrete component for $G = SO(n, 1; \mathbb{K})$, $H = SO(n - 1, 1; \mathbb{K})$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ being the fields of real, complex, quaternion numbers, or for $G = F_{4(-20)}$ and $H = Spin(8, 1) \subset G$. We shall use the compact realization of the spherical principal series π_ν on the sphere $S = K/M$ in \mathbb{F}^n . We prove that for appropriate small parameter

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ν the natural restriction map of functions on S in π_ν to the lower dimensional sphere S^\flat in \mathbb{F}^{n-1} defines a bounded operator onto a complementary series π_ν^\flat of H . The proof requires rather detailed study of the restriction to $S^\flat \subset S$ of spherical harmonics on S .

The representations π_ν for certain integers ν have also unitarizable subquotients or subrepresentations. Some of them are discrete series for $SU(2, 1)$. We shall find irreducible components for the representations under the subgroup H . One easiest case is the subrepresentation π_0^\pm (or π_{2n+2}^\pm as quotient) of the group $SU(n, 1)$. The space π_0^\pm consists of holomorphic respectively antiholomorphic polynomials on \mathbb{C}^n modulo constant functions. It can also be treated by using the analytic continuation of scalar holomorphic discrete series at the reducible point [7], and some general decomposition results have been obtained in [16]. These representations are also of special interests in automorphic representation theory [21].

We note that our result can be understood heuristically as certain boundedness property of the restriction map from certain Sobolev spaces on S to those on S^\flat . Indeed for small parameter ν the space π_ν consists of distributions on S whose fractional differentiations are in $L^2(S)$, i.e., they are functions with certain smooth conditions. It is thus expected that their restriction on the subsphere S^\flat would make sense in proper Sobolev spaces. A precise formulation can be done and we hope to return to it in future. We remark also that the study of the norm estimates of the restriction of the spherical harmonics on lower dimensional spheres can be put into a general context as the study of growth of L^p -norm of restriction on totally geodesic submanifolds of eigenstates of Laplace-Beltrami operators on Riemannian manifold; see [5]. Our results here give precise estimates of the L^2 -norm of the restriction. It would be interesting to see if there are similar type of results for general compact manifolds of positive curvature.

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2. PRELIMINARIES

2.1. Classical rank one groups. Let $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the real, complex and quaternionic numbers. Denote $G := SO_0(n, 1; \mathbb{F}) = SO_0(n, 1), SU(n, 1), Sp(n, 1)$ the connected component of group of \mathbb{F} -linear transformations on \mathbb{F}^{n+1} preserving the quadratic form $|x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$, with \mathbb{F} acting on the right. The group $K := SO_0(n), S(U(n) \times U(1)), Sp(n) \times Sp(1)$ is a maximal compact subgroup of G and G/K is a Riemannian symmetric space of rank one which can further be realized as the unit ball in \mathbb{F}^n . Elements in G and \mathfrak{g} will be written as $(n+1) \times (n+1)$ block \mathbb{F} -matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are of size $n \times n, n \times 1, 1 \times n, 1 \times 1$, respectively.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. We fix

$$H_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

in \mathfrak{p} and let $\mathfrak{a} = \mathbb{R}H_0 \subset \mathfrak{p}$. Then \mathfrak{a} is a maximal abelian subspace of \mathfrak{p} . The root space decomposition of \mathfrak{g} under H_0 is

$$\mathfrak{g} = \mathfrak{g}_{-1} + (\mathfrak{a} + \mathfrak{m}) + \mathfrak{g}_1$$

with roots $\pm 1, 0$ if $\mathbb{F} = \mathbb{R}$, and

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + (\mathfrak{a} + \mathfrak{m}) + \mathfrak{g}_1 + \mathfrak{g}_2$$

with roots $\pm 2, \pm 1, 0$, if $\mathbb{F} = \mathbb{C}, \mathbb{H}$. Here $\mathfrak{m} \subset \mathfrak{k}$ is the zero root space in \mathfrak{k} . We denote

$$\mathfrak{n} = \mathfrak{g}_1, \quad \mathfrak{n} = \mathfrak{g}_1 + \mathfrak{g}_2$$

the sum of the positive root spaces, in the respective cases. Thus $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is a maximal parabolic subalgebra of \mathfrak{g} . Let ρ be the half sum of positive roots. With some abuse of notation we write $\rho(H_0) = \rho$, and we have

$$(2.1) \quad \rho = \begin{cases} \frac{n-1}{2}, & \mathbb{F} = \mathbb{R} \\ n, & \mathbb{F} = \mathbb{C} \\ 2n+1, & \mathbb{F} = \mathbb{H}. \end{cases}$$

Denote M, A, N the corresponding subgroups with Lie algebras $\mathfrak{m}, \mathfrak{a}, \mathfrak{n}$. Then $M = SO_0(n-1), SU(n-1), Sp(n-1) \times Sp(1)$ and MAN is a maximal parabolic subgroup of G .

2.2. Decomposition of $L^2(K/M)$. We identify \mathfrak{p} with \mathbb{F}^n and normalize the K -invariant product on \mathfrak{p} so that H_0 is a unit vector. The homogeneous space K/M is then the unit sphere $S := S^{dn-1}$ in $\mathfrak{p} = \mathbb{F}^n$ with M being the isotropic subgroup of the base point $H_0 \in \mathfrak{p}$. We denote dx the normalized area measure on S and $L^2(S)$ the corresponding L^2 -space. For $n = 1$ the decomposition of $L^2(K/M)$ is well-known and elementary, so we assume $n > 1$. Let W^p be the space of spherical harmonics on S . For $\mathbb{F} = \mathbb{C}$ let $W^{p,q}$ be the spherical harmonics of degree $p+q$ on \mathbb{C}^n and holomorphic of degree p and antiholomorphic of degree q . If $\mathbb{F} = \mathbb{H}$, then $K = Sp(n) \times Sp(1)$, and its representations are of the form $\tau_1 \times \tau_1$, which will be written as (τ_1, τ_2) and further identified with their highest weights. The root system of $Sp(n)$ is of type C and let $\alpha_1, \dots, \alpha_{n-1}, \alpha_n$ be the simple roots with α_n the longest one. Denote $\lambda_1, \dots, \lambda_n$ the corresponding fundamental weights with λ_1 the defining representation on \mathbb{C}^{2n} . For $Sp(1) = SU(2)$ the representation on symmetric tensor power $\odot^q(\mathbb{C}^2) = \mathbb{C}^{q+1}$ will be written just as q for simplicity. Denote $W^{p,q}$ the representation $(q\lambda_1 + \frac{p-q}{2}\lambda_2, q)$ of $K = Sp(n) \times Sp(1)$.

Recall [17, 12]

$$(2.2) \quad L^2(S) = \sum_{\tau}^{\oplus} W^{\tau}, \quad W^{\tau} = \begin{cases} W^p, p \geq 0 & \mathbb{F} = \mathbb{R} \\ W^{p,q}, p, q \geq 0 & \mathbb{F} = \mathbb{C} \\ W^{p,q}, p \geq q \geq 0, p - q \text{ even} & \mathbb{F} = \mathbb{H} \end{cases}$$

Here and in the following we denote a general representation of K by τ . The subspace $(W^{\tau})^M$ of M -fixed vector is one dimensional

$$(W^{\tau})^M = \mathbb{C}\phi_{\tau}$$

where ϕ_{τ} is normalized by $\phi_{\tau}(H_0) = 1$. They depend only on the first variable $x_1 \in \mathbb{H}$ of $x = (x_1, \dots, x_n)$, and will also be written as $\phi_{\tau}(x_1)$. They are given by

$$(1) \quad \mathbb{F} = \mathbb{R}, x_1 = \cos \xi,$$

$$\phi_p^n(x_1) := \cos^p \xi F\left(-\frac{p}{2}, -\frac{p-1}{2}, \frac{n-1}{2}, -\tan^2 \xi\right);$$

$$(2) \quad \mathbb{F} = \mathbb{C}, x_1 = e^{it} \cos \xi,$$

$$\phi_{p,q}^n(x_1) = e^{it(p-q)} \cos^{p+q} \xi F(-p, -q, n-1, -\tan^2 \xi);$$

$$(3) \quad \mathbb{F} = \mathbb{H}, x_1 = \cos \xi e^{ty} = \cos \xi (\cos t + y \sin t) \text{ in quaternionic polar coordinates, } y \text{ being purely imaginary and } |y| = 1,$$

$$\phi_{p,q}^n(x) = \phi_{p,q}^n(x_1) := \frac{\sin(q+1)t}{\sin t} \cos^p \xi F\left(-\frac{p-q}{2}, -\frac{p+q+2}{2}, 2(n-1), -\tan^2 \xi\right).$$

See [12, Theorem 3.1]. (Note that in the formula for $\psi_{p,q}$ and $e_{p,q}$ in [12, p.144-147] the term $\frac{-p-q}{2}$ should be $\frac{-p+q}{2}$.) Here $F(a, b, c, x)$ is the Gauss hypergeometric function ${}_2F_1$,

$$F(a, b, c, x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}$$

and $(a)_m = \prod_{j=0}^{m-1} (a+j)$ is the Pochhammer symbol. Note that all ϕ -functions above are Jacobi polynomials [24] in $t = 2x_1^2 - 1$ in the interval $(-1, 1)$.

We put the upper-index the dimension n as we shall also treat it as a variable.

In particular we have, by Schur's orthogonality relation,

$$(2.3) \quad \|\phi_{\tau}\|^2 = \frac{1}{\dim(W^{\tau})}.$$

$\dim(W^{\tau})$ can be evaluated by the Weyl's dimension formula: Let $\{\alpha\}$ be the root system of \mathfrak{k} with $\{\alpha > 0\}$ the positive roots and $\rho_{\mathfrak{k}}$ the half sum of the positive roots,

$$\dim(W^{\tau}) = \prod_{\alpha > 0} \frac{\langle \tau + \rho_{\mathfrak{k}}, \alpha \rangle}{\langle \rho_{\mathfrak{k}}, \alpha \rangle};$$

see e.g. [9].

We shall also need a general integral formula: If $f(x) = g(y)h(z)$, $x = (y, z)$ are functions on \mathbb{R}^m with separated variables $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{m-k}$ with dy the Lebesgue

measure then we have

$$(2.4) \quad \int_{S^{m-1}} f(x) dx = \frac{2\Gamma(\frac{m}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{m-k}{2})\omega_{k-1}} \int_{B^k} g(y)(1-|y|^2)^{\frac{1}{2}(m-k-2)} \\ \times \left(\int_{S^{m-k-1}} h((1-|y|^2)^{\frac{1}{2}}z) dz \right) dy$$

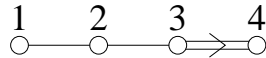
where dx and dz are the normalized area measures on the respective spheres and $\omega_{k-1} = \frac{2\sqrt{\pi}^k}{\Gamma(\frac{k}{2})}$ is the Lebesgue area of the sphere in \mathbb{R}^k (we shall need $k = 1, 2, 4$ only); see e.g. [20, 1.4.4 (1)] for the case of even m and k . Thus the square norm $\|\phi_\tau\|^2$ can also be proved by using the known integral formulas for Jacobi polynomials. However we shall use mostly the Weyl's dimension formula whenever possible as it is conceptionally clearer and as their asymptotic are well-understood.

2.3. Exceptional group $F_{4(-20)}$. Let G be the connected Lie group of type $F_{4(-20)}$ with Lie algebra \mathfrak{g} . This group has been well-studied [11, 26]. The maximal compact subgroup K is $Spin(9)$ and the symmetric space G/K can be realized as the unit ball in \mathbb{O}^2 with \mathbb{O} being the Cayley division (octonian) algebra. Let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition. The space \mathfrak{p} will be identified with \mathbb{O}^2 with $\mathfrak{k} = spin(9)$ acting on \mathbb{O}^2 via the Spin representation. We fix $H_0 \in \mathbb{O}^2 = \mathfrak{p}$ so that the positive roots of H_0 in \mathfrak{g} are 2, 1. The corresponding multiplicities are then 7 and 8. The half sum of positive roots is $\rho = 11$. Let \mathfrak{m} be the zero root space of H_0 in \mathfrak{k} , and $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ the maximal parabolic subalgebra.

The algebra $\mathfrak{m} \subset \mathfrak{k}$ is $spin(7)$. Let $M = Spin(7)$ be the corresponding simply connected subgroup with Lie algebra \mathfrak{m} . Fix the K -invariant inner product on $\mathfrak{p} = \mathbb{O}^2$ with H_0 being unit vector. The homogeneous space K/M is the unit sphere $S = S^{15}$ in $\mathbb{O}^2 = \mathbb{R}^{16}$. To describe the decomposition of $L^2(S)$ under K we observe first that the space $\mathfrak{p} = \mathbb{O}^2$ is decomposed under M as

$$(2.5) \quad \mathfrak{p} = \mathbb{O} \oplus \mathbb{O} = (\mathbb{R}H_0 \oplus \mathbb{R}^7) \oplus \mathbb{O}$$

with \mathbb{R}^7 being the defining representation of $SO(7)$ and thus of M via the double covering $M = Spin(7) \rightarrow SO(7)$, and \mathbb{O} the Spin representation of M . The Dynkin diagram of $Spin(9)$ is



with the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the corresponding fundamental weights. Let $W^{p,q}$ be the representation of K with highest weight $\frac{p-q}{2}\lambda_1 + q\lambda_4$. Then it follows [17, 11] that

$$(2.6) \quad L^2(S) = \bigoplus_{p \geq q \geq 0, p-q=0 \text{ even}} W^{p,q},$$

and each space $W^{p,q}$ has a unique M -fixed vector $\phi_{p,q}$, $(W^{p,q})^M = \mathbb{C}\phi_{p,q}$, such that $\phi_{p,q}(H_0) = 1$. To describe $\phi_{p,q}$ write elements in \mathbb{O}^2 as $x = (x_0, x_1, x_2)$ under the decomposition (2.5), and write their (partial) polar coordinates as $r = |x|$, $\sqrt{x_0^2 + \|x_1\|^2} = r \cos \xi$, $x_0 = r \cos \xi \cos \eta$ with $0 \leq \xi \leq \frac{\pi}{2}$, $0 \leq \eta \leq \pi$. Then

$$\begin{aligned} \phi_{p,q}(x) &= \phi_{p,q}(x_0) \\ &= \cos^q \eta F\left(-\frac{q}{2}, -\frac{q-1}{2}, \frac{7}{2}; -\tan^2 \eta\right) \cos^p \xi F\left(-\frac{p-q}{2}, -\frac{p+q+6}{2}, 4; -\tan^2 \xi\right), \end{aligned}$$

for $x \in S$; see [11].

3. RESTRICTION OF $(SO_0(n, 1; \mathbb{F}), \pi_\nu)$ TO $(SO_0(n-1, 1; \mathbb{F}))$

3.1. Principal series of G . For $\nu \in \mathbb{C}$ let π_ν be the induced representation of G from MAN consisting of measurable functions f on G such that

$$(3.1) \quad f(gme^{tH_0}n) = e^{-\nu t} f(g), e^{tH_0}mn \in MAN$$

and $f|_K \in L^2(K)$. (Our representation π_ν is $Ind_{MAN}^G(e^{\rho+\nu})$ in the standard notation [14]. However the parameter ν has some advantage it is “stable” under branching; see Theorem 3.6 below.) In particular f in π_ν are invariant under M , and π_ν is further realized on $L^2(K/M) = L^2(S)$. We denote X_ν the corresponding (\mathfrak{g}, K) -module with X_ν the algebraic sum of K -irreducible subspaces in $L^2(K/M)$. To simplify notation we shall denote π_ν also the corresponding unitary representation of G when $(X_\nu, \pi_\nu, \mathfrak{g})$ is unitarizable.

The L^2 -norm in $L^2(S)$ is not unitary for π_ν except when $\nu = \rho + it$ for $t \in \mathbb{R}$, ρ being given by (2.1). The unitarizable representations $(X_\nu, \pi_\nu, \mathfrak{g})$ for real ν are usually called complementary series. They have been found in [12]. (See also [6] for related results for the real group $SO_0(n, 1)$.) The constant $\lambda_\nu(\tau)$ below are rewritten in terms of the Pochhammer symbol $(a)_m$ and further the Gamma functions.

Theorem 3.1. *There is a positive definite \mathfrak{g} -invariant form on X_ν given by*

$$\|w\|_\nu^2 = \lambda_\nu(\tau) \|w\|^2$$

and its completion forms an unitary irreducible representation of G , if

$$(1) \quad \mathbb{F} = \mathbb{R}, 0 < \nu < n-1,$$

$$(3.2) \quad \lambda_\nu(p) = \frac{(n-1-\nu)_p}{(\nu)_p} = \frac{\Gamma(n-1-\nu+p)}{\Gamma(n-1-\nu)\Gamma(\nu+p)};$$

$$(2) \quad \mathbb{F} = \mathbb{C}, 0 < \nu < 2n,$$

$$(3.3) \quad \lambda_\nu(p, q) = \frac{(n-\frac{\nu}{2})_p (n-\frac{\nu}{2})_q}{(\frac{\nu}{2})_p (\frac{\nu}{2})_q} = \frac{\Gamma^2(\frac{\nu}{2})\Gamma(n-\frac{\nu}{2}+p)\Gamma(n-\frac{\nu}{2}+q)}{\Gamma^2(n-\frac{\nu}{2})\Gamma(\frac{\nu}{2}+p)\Gamma(\frac{\nu}{2}+q)};$$

(3) $\mathbb{F} = \mathbb{H}$, $2 < \nu < 4n$,

$$(3.4) \quad \begin{aligned} \lambda_\nu(p, q) &= \frac{(2n - \frac{\nu}{2})^{\frac{p-q}{2}} (2n + 1 - \frac{\nu}{2})^{\frac{p+q}{2}}}{(\frac{\nu}{2} - 1)^{\frac{p-q}{2}} (\frac{\nu}{2})^{\frac{p+q}{2}}} \\ &= \frac{\Gamma(\frac{\nu}{2} - 1) \Gamma(\frac{\nu}{2})}{\Gamma(2n - \frac{\nu}{2}) \Gamma(2n + 1 - \frac{\nu}{2})} \frac{\Gamma(2n - \frac{\nu}{2} + \frac{p-q}{2}) \Gamma(2n + 1 - \frac{\nu}{2} + \frac{p+q}{2})}{\Gamma(\frac{\nu}{2} - 1 + \frac{p-q}{2}) \Gamma(\frac{\nu}{2} + \frac{p+q}{2})}. \end{aligned}$$

3.2. General criterion of boundedness. We fix n and let $H = SO_0(n-1, 1; \mathbb{F}) \subset G$ be the subgroup fixing the n -th coordinate x_n in \mathbb{F}^{n+1} . Denote $L := K \cap H$, a maximal subgroup of H . The subsphere $S^{d(n-1)-1}$ in \mathbb{F}^{n-1} of the sphere $S = K/M \subset \mathbb{F}^n$ defined by the equation $x_n = 0$ will be written as S^b , which is homogeneous space of L , $S^b = L/L \cap M$. To avoid confusion we denote by π_ν^b the corresponding representations of H and X_ν^b the L -finite vectors, and the corresponding decomposition of $L^2(S^b) = L^2(L/L \cap M)$ will be written as

$$L^2(S^b) = \sum_{\sigma}^{\oplus} V^{\sigma}$$

with σ being specified accordingly.

We shall need a general and elementary criterion for boundedness of intertwining operators. The sufficient part of the following Lemma 3.2 is used in [22] implicitly, and we give here a proof for the sake of completeness. Let K be a temporarily a compact group and $L \subset K$ a closed subgroup. Let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ and $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be two Hilbert spaces invariant under K and respectively L . Consider

$$\mathcal{W}|_K = \sum_{\tau}^{\oplus} W^{\tau}, \quad \mathcal{V}|_L = \sum_{\sigma}^{\oplus} V^{\sigma}$$

the irreducible decomposition of \mathcal{W} and \mathcal{V} under K and respectively L counting multiplicities, all assumed being finite. Consider further the branching of W^{τ} under L . Write $\sigma \subset \tau$ if a representation σ appears in τ (counting multiplicities) with $\tilde{V}^{\tau, \sigma}$ the corresponding irreducible component, and denote $P_{\tau, \sigma}$ the corresponding orthogonal projection, i.e.,

$$W^{\tau} = \sum_{\sigma \subset \tau}^{\oplus} \tilde{V}^{\tau, \sigma}, \quad P_{\tau, \sigma} : W^{\tau} \rightarrow \tilde{V}^{\tau, \sigma}.$$

Suppose R is a densely defined L -invariant operator from K -finite elements in \mathcal{W} to L -finite elements in \mathcal{V} , and

$$R_{\tau, \sigma} := P_{\tau, \sigma} R : W^{\tau} \rightarrow V^{\sigma}$$

its components, i.e., $R = \sum_{\tau} \sum_{\sigma \subset \tau} R_{\tau, \sigma}$ on K -finite functions. We write $\|R\|_{\mathcal{W}, \mathcal{V}}$ its norm whenever it is finite.

Lemma 3.2. *The restriction operator R extends to a bounded operator from \mathcal{W} to \mathcal{V} if and only if there is a constant C such that for any σ*

$$(3.5) \quad \sum_{\tau \supset \sigma} \|R_{\tau,\sigma}\|_{\mathcal{W},\mathcal{V}}^2 \leq C.$$

Proof. Let $w = \sum_{\tau} w_{\tau} \in \mathcal{W}$ be an element with finite many nonzero components w_{τ} . Its squared norm in \mathcal{W} is

$$\|w\|_{\mathcal{W}}^2 = \sum_{\tau} \|w_{\tau}\|_{\mathcal{W}}^2$$

by our assumption. We compute the norm $\|Rw\|_{\mathcal{V}}$. Writing w as $w = \sum_{\tau} w_{\tau} = \sum_{\tau} \sum_{\sigma \subset \tau} P_{\tau,\sigma} w_{\tau}$, we have

$$Rw = \sum_{\sigma} \sum_{\tau \supset \sigma} R_{\tau,\sigma} P_{\tau,\sigma} w_{\tau},$$

and

$$\|Rw\|_{\mathcal{V}}^2 = \sum_{\sigma} \left\| \sum_{\tau \supset \sigma} R_{\tau,\sigma} P_{\tau,\sigma} w_{\tau} \right\|_{\mathcal{V}}^2 \leq \sum_{\sigma} \left(\sum_{\tau \supset \sigma} \|R_{\tau,\sigma}\|_{\mathcal{W},\mathcal{V}} \|P_{\tau,\sigma} w_{\tau}\|_{\mathcal{W}} \right)^2.$$

If the condition (3.5) is satisfied we find, by Cauchy-Schwarz inequality, that

$$\|Rw\|_{\mathcal{V}}^2 \leq \sum_{\sigma} \left(\sum_{\tau \supset \sigma} \|R_{\tau,\sigma}\|_{\mathcal{W},\mathcal{V}}^2 \right) \left(\sum_{\tau \supset \sigma} \|P_{\tau,\sigma} w_{\tau}\|_{\mathcal{W}}^2 \right)$$

which is dominated by

$$C \sum_{\sigma} \sum_{\tau \supset \sigma} \|P_{\tau,\sigma} w_{\tau}\|_{\mathcal{W}}^2 = C \sum_{\tau} \sum_{\sigma \subset \tau} \|P_{\tau,\sigma} w_{\tau}\|_{\mathcal{W}}^2 = C \sum_{\tau} \|w_{\tau}\|_{\mathcal{W}}^2 = C \|w\|_{\mathcal{W}}^2,$$

finishing the proof of sufficiency. Conversely suppose R is a bounded operator. Then so is R^* , and for a given $v \in V^{\sigma}$ we have

$$C \|v\|_{\mathcal{V}}^2 \geq \|R^* v\|_{\mathcal{W}}^2 = \left\| \sum_{\tau \supset \sigma} R_{\sigma,\tau}^* v \right\|_{\mathcal{W}}^2 = \sum_{\tau \supset \sigma} \|R_{\sigma,\tau}^* v\|_{\mathcal{W}}^2.$$

But each $R_{\sigma,\tau}^*$ is a scalar constant of an isometric operator by Schur's lemma, and we have

$$\|R_{\sigma,\tau}^* v\|_{\mathcal{W}}^2 = \|R_{\sigma,\tau}^*\|_{\mathcal{V},\mathcal{W}}^2 \|v\|_{\mathcal{V}}^2 = \|R_{\tau,\sigma}\|_{\mathcal{W},\mathcal{V}}^2 \|v\|_{\mathcal{V}}^2.$$

Substituting this into the above inequality we obtain (3.5). \square

3.3. Restriction of spherical harmonics. We specify the above considerations to the restriction $R : C^{\infty}(S) \rightarrow C^{\infty}(S^{\flat})$, $f(x', x_n) \rightarrow f(x')$. The branching of $W^{\tau} = \sum_{\sigma} \tilde{V}^{\tau,\sigma}$ under L can be read off abstractly from known results. However we need to find all isotypic L -irreducible subspaces $\tilde{V}^{\tau,\sigma} \subset W^{\tau}$ with nonzero restriction, i.e. with the restriction

$$R_{\tau,\sigma} : \tilde{V}^{\tau,\sigma} \rightarrow V^{\sigma}$$

acting as an isomorphism. We shall drop the upper-index τ in $\tilde{V}^{\tau,\sigma}$ in the lemma below, as it is fixed in the summation.

Lemma 3.3. (1) $\mathbb{F} = \mathbb{R}$. The branching of W^p under $L = SO(n-1)$ is multiplicity free. The restriction $W^p|_{x_n=0}$ under $L = SO(n-1)$ is decomposed as

$$W^p|_{x_n=0} = \sum_{0 \leq q \leq p, p-q \text{ even}}^{\oplus} V^q$$

The corresponding unique q -isotypic component in W^p is given by (as functions on S)

$$\tilde{V}^q = \{h(x')\phi_{p-q}^{n+2q}(x_n); h \in V^q\}$$

(2) $\mathbb{F} = \mathbb{C}$. The branching of $W^{p,q}$ under $L = U(n-1)$ is multiplicity free. The space $W^{p,q}|_{x_n=0}$ under L is decomposed as

$$W^{p,q}|_{x_n=0} = \sum_{p_1 \leq p, q_1 \leq q, p-p_1=q-q_1}^{\oplus} V^{p_1, q_1}.$$

For each (p_1, q_1) the unique (p_1, q_1) -isotypic component in $W^{p,q}$ is given by

$$\tilde{V}^{p_1, q_1} = \{h(x')\phi_{p-p_1, q-q_1}^{n+p_1+q_1}(x_n); h \in V^{p_1, q_1}\}.$$

(3) $\mathbb{F} = \mathbb{H}$. The space $W^{p,q}|_{x_n=0}$ under $L = Sp(n-l) \times Sp(1)$ is decomposed as

$$W^{p,q}|_{x_n=0} = \sum_{p \geq p_1, q_1=q}^{\oplus} V^{p_1, q_1}.$$

The corresponding (p_1, q_1) -isotypic component is given by

$$\tilde{V}^{p_1, q_1} = \{h(z')\phi_{p-p_1, 0}^{n+\frac{p_1}{2}}(x_n); h \in V^{p_1, q_1}\}$$

Proof. Let $\mathbb{F} = \mathbb{R}$. The multiplicity free result in this case is well-known. The statement on the restriction is a result of Vilenkin [27, (9), p. 495]. The proof there relies on explicit computations for the projection into spherical harmonics, which seem not easy to generalize to other cases. We give a slightly different proof which applies also to the other cases and which avoids some redundant computations. Denote $L_n = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ the Laplacian on \mathbb{R}^n . Recall that the spherical polynomial $f = r^m C_m^{\frac{n-2}{2}}(\frac{x_n}{r})$ is the unique $SO(n-1)$ invariant polynomials on \mathbb{R}^n of degree m satisfying $L_n f = 0$, where $C_m^{\frac{n-2}{2}}(t)$ is the Gegenbauer polynomial. Let $x = (x', x_n) \in \mathbb{R}^n$, and put $u := |x'|$, $v := x_n$. We have

$$L_n = L_{n-1} + \frac{\partial^2}{\partial v^2} = \frac{\partial^2}{\partial u^2} + \frac{n-2}{u} \frac{\partial}{\partial u} + \frac{\partial^2}{\partial v^2},$$

when acting functions depending only on $|x'|$ and x_n . Rephrasing in terms of u, v we have the unique solution of the form $f(u, v) = (u^2 + v^2)^{\frac{m}{2}} C(\frac{v}{\sqrt{u^2+v^2}})$, of the equation

$$(3.6) \quad L_n f = \frac{\partial^2 f}{\partial u^2} + \frac{n-2}{u} \frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial v^2} = 0.$$

is when $C = C_m^{\frac{n-2}{2}}$. Now for fixed $q \leq p$ we search for an isotypic $SO(n-1)$ -component in W^p of type V^q consisting of homogeneous polynomials $F(x)$ of degree p of the form

$F(x) = h(x')f(u, v) = h(x')f(|x'|, x_n)$, where h is a spherical harmonics of degree q on \mathbb{R}^{n-1} , i.e. $L_{n-1}h = 0$, and $f(u, v) = (u^2 + v^2)^{\frac{p-q}{2}} C(\frac{v}{\sqrt{u^2+v^2}})$. The Laplace equation $L_n F = (L_{n-1} + \frac{\partial^2}{\partial v^2})F = 0$ becomes

$$(L_{n-1}h(x'))f(u, v) + 2 \sum_{j=1}^{n-1} x_j \frac{\partial h(x')}{\partial x_j} \frac{1}{u} \frac{\partial h}{\partial u} f(u, x_n) + h(x') L_n f(u, x_n) = 0,$$

with $L_n f$ computed in (3.6). But $L_{n-1}h(x') = 0$ and $\sum_{j=1}^{n-1} x_j \frac{\partial h(x')}{\partial x_j} = qh(x')$ by our assumption. Thus it reduces to

$$(3.7) \quad 2q \frac{1}{u} \frac{\partial h}{\partial u} f(u, v) + L_n f(u, v) = 0,$$

i.e.

$$(3.8) \quad \frac{\partial^2 f}{\partial u^2} + \frac{n-2+2q}{u} \frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial v^2} = 0.$$

This is precisely the equation (3.6) with n replaced by $n+2q$ and m replaced by $p-q$. Thus f is a constant multiple of $r^{p-q} C_q^{\frac{n-2}{2}+q}(\frac{x_n}{r})$ (which is a posterior polynomial in x). This proves the case for $\mathbb{F} = \mathbb{R}$.

$\mathbb{F} = \mathbb{C}$. The multiplicity free result is also known; see e.g. [13]. The abstract decomposition of $W^{p,q}|_{z_n=0}$ follows easily by counting the degrees (p_1, q_1) . The same argument above applies and we get that the polynomials of the type $F(x) = h(x')r^{p+q-p_1-q_1} C(\frac{x_n}{r})$ as stated are spherical harmonics. Moreover they are of the right type (p, q) and thus they are in the space $W^{p,q}$.

$\mathbb{F} = \mathbb{H}$. The group $Sp(1)$ acts on the space of polynomials on the right, $h \in Sp(1) : f(x) \mapsto f(xh)$, and it acts on the space $W^{p,q}$ as the symmetric tensor $\odot^q(\mathbb{C}^2)$. So does it also on the space $W^{p,q}|_{x_n=0}$. Thus any irreducible component must be of type V^{p_1, q_1} with $q_1 = q$, again by (2.1). In particular $p - p_1 = (p - q) - (p_1 - q_1)$ is even since both $p - q$ and $p_1 - q_1$ are even. This proves the decomposition. The rest of the proof is almost the same as above. (Note that the function $\phi_{p-p_1, 0}^{n+\frac{p_1}{2}}$ is obtained from $\phi_{p, 0}^n$ in §2.2 by formally replacing n by $n + \frac{p_1}{2}$, which is not necessarily an integer.) \square

We compute now the norm of $R_{\tau, \sigma}$. For positive constants $C_{\tau, \sigma}$ and $D_{\tau, \sigma}$ we write $C_{\tau, \sigma} \sim D_{\tau, \sigma}$ if both $\frac{C_{\tau, \sigma}}{D_{\tau, \sigma}}$ and $\frac{D_{\tau, \sigma}}{C_{\tau, \sigma}}$ are dominated by positive constants independent of τ, σ .

Proposition 3.4. *With the notation as above we have the norm of $R_{\tau, \sigma} : W^\tau \subset L^2(S) \rightarrow V^\sigma \subset L^2(S^b)$ is given by*

$$\begin{aligned} (1) \quad & \mathbb{F} = \mathbb{R}, p - q \geq 0 \text{ even}, \\ \|R_{p,q}\|^2 &= \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \frac{(2p+n-2)\Gamma(\frac{n+p+q-2}{2})\Gamma(\frac{p-q+1}{2})}{\Gamma(\frac{p-q+2}{2})\Gamma(\frac{n+p+q-1}{2})} \sim \frac{p+1}{(p+q+1)^{\frac{1}{2}}(p-q+1)^{\frac{1}{2}}}; \\ (2) \quad & \mathbb{F} = \mathbb{C}, p \geq p_1 \geq 0, p_1 \geq q_1 \geq 0, p - q = p_1 - q_1, \\ & \|R_{(p,q),(p_1,q_1)}\|^2 = p + q + n - 1; \end{aligned}$$

(3) $\mathbb{F} = \mathbb{H}$, $p - p_1 = 2k \geq 0$ and $p_1 - q_1 \geq \text{even}$, $q = q_1$,

$$\begin{aligned} \|R_{(p,q),(p_1,q)}\|^2 &= \frac{\Gamma(2n-2)}{\Gamma(2n)} (k+1)(2k+2(n-1)+p_1-1)(k+2(n-1)+p_1) \\ &\sim (k+1)(k+p_1+1)^2. \end{aligned}$$

In all other cases of (τ, σ) we have $R_{\tau,\sigma} = 0$.

Proof. $\mathbb{F} = \mathbb{R}$. By the previous lemma and Schur lemma we see that $R_{p,q} : W^p \rightarrow V^q$ is up to a constant a partial isometry, and $\widetilde{V}^q \rightarrow V^q$ is up to a constant an isometry. Thus

$$\|R_{p,q}\|^2 = \frac{\|Rf\|^2}{\|f\|^2}$$

for any $0 \neq f \in \widetilde{V}^q$. Now we take $f = h(x')\phi_{p-q}^{n+2q}(x_n)$, which has a form of variable separation, and we have, by (2.4) and the q -homogeneity of $h(x')$, that

$$\|f\|^2 = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+q} |\phi_{p-q}^{n+2q}(x_n)|^2 \int_{S^b} |h(y')|^2 dy' dx_n,$$

and

$$(3.9) \quad \|Rf\|^2 = |\phi_{p-q}^{n+2q}(0)|^2 \int_{S^b} |h(y')|^2 dy'.$$

Consequently

$$\|R_{p,q}\|^2 = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} |\phi_{p-q}^{n+2q}(0)|^2 \left(\int_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+q} |\phi_{p-q}^{n+2q}(x_n)|^2 dx_n \right)^{-1}.$$

Note that the integral $I := \int_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+q} |\phi_{p-q}^{n+2q}(x_n)|^2 dx_n$ is up to a constant the square norm in $L^2(S^{n+2q-1})$ of the spherical polynomial $\phi_{p-q}^{n+2q}(x_n)$ in dimension $n+2q$, and can be evaluated by using (2.3) in terms of the dimension $\dim W_{n+2q}^{p-q}$ of the representation of $SO(n+2q)$. The exact (a rather subtle) constant is computed in (2.4),

$$\int_{|x_n|<1} (1 - |x_n|^2)^{\frac{n-3}{2}+q} |\phi_{p-q}^{n+2q}(x_n)|^2 dx_n = \frac{\Gamma(\frac{n+2q-1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{n+2q}{2})} \frac{1}{\dim W_{n+2q}^{p-q}}.$$

Thus using the dimension formula that

$$\dim W_n^j = \binom{n+j-1}{j} - \binom{n+j-3}{j-2} = \frac{(n+2j-2)\Gamma(n+j-2)}{\Gamma(j+1)\Gamma(n-1)},$$

we have I^{-1} is

$$\frac{2\Gamma(\frac{n+2q}{2})}{\Gamma(\frac{n+2q-1}{2})\Gamma(\frac{1}{2})} \frac{(2p+n-2)\Gamma(n+p+q-2)}{\Gamma(n+2q-1)\Gamma(p-q+1)}.$$

The evaluation $\phi_{p-q}^{n+2q}(0)$ in (3.9) is zero unless $p - q = 2k$ is even, in which case it is

$$(-1)^k \frac{(-k)_k (-\frac{p-q-1}{2})_k}{(\frac{n+2q-1}{2})_k k!}.$$

But $(-k)_k = (-1)^k k!$, $(-\frac{p-q-1}{2})_k = (-1)^k (\frac{1}{2})_k = (-1)^k \frac{\Gamma(\frac{1}{2}+k)}{\Gamma(\frac{1}{2})}$, we find that the evaluation, disregarding the sign $(-1)^k$ and the constant $\Gamma(\frac{1}{2})$, is

$$\frac{\Gamma(\frac{p-q+1}{2})\Gamma(\frac{n+2q-1}{2})}{\Gamma(\frac{n+p+q-1}{2})}.$$

Using the product formula $\Gamma(2x) = \Gamma(\frac{1}{2})^{-1} 2^{2x-1} \Gamma(x) \Gamma(x + \frac{1}{2})$ we obtain then the formula for $\|R_{p,q}\|^2$ as stated. The rest follows from the Stirling formula that

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \sim n^{a-b}, \quad n \rightarrow \infty.$$

The case $\mathbb{F} = \mathbb{C}$ is done by similar computations. In the case $\mathbb{F} = \mathbb{H}$ we have

$$\|R_{(p,q),(p_1,q)}\|^2 = \frac{|\phi_{p-p_1}^{n+\frac{p_1}{2}}(0)|^2 \int_{S^b} |h(x')|^2 dx'}{\|\phi_{p-p_1}^{n+\frac{p_1}{2}} h\|^2}$$

with $\|\phi_{p-p_1}^{n+\frac{p_1}{2}} h\|^2$ being

$$\frac{2\Gamma(n)}{\Gamma(n-1)\omega_3} \int_{x_n \in \mathbb{H}, |x_n| < 1} |\phi_{p-p_1}^{n+\frac{p_1}{2}}(x_n)|^2 (1 - |x_n|^2)^{\frac{1}{2}(4(n-1)-2+2p_1)} \int_{S^b} |h(x')|^2 dx' dx_n$$

by the integral formula above for separated variables. The norm of $\phi^{n+\frac{p_1}{2}}$ can not be computed using the dimension formula for p_1 odd as it can not be interpreted as spherical polynomials on a symmetric space. However we may use by known integral formulas [24] for Jacobi polynomials $P^{(\alpha,\beta)}(t)$ on the interval $[-1, 1]$. (More generally one may use the theory of Heckman-Opdam [8] for Jacobi polynomials with general root multiplicities.) Indeed the function $\phi_{k,0}^n$ in §2.2 for any real $n > 1$ can be written as

$$\phi_{k,0}^n(x) = \frac{\Gamma(k+1)\Gamma(2n-2)}{\Gamma(k+2n-2)} P_k^{(2n-3,1)}(2|x|^2-1)$$

where $|x|$ is the norm of a quaternionic number $x \in \mathbb{H}$. The norm to be computed is

$$\int_{x \in \mathbb{H}, |x| < 1} |\phi_{m,0}^m(x)|^2 (1 - |x|^2)^{\frac{1}{2}(4(n-1)-2)} dx = \omega_3 \int_0^1 |\phi_{m,0}^m(x)|^2 (1 - |x|^2)^{\frac{1}{2}(4(n-1)-2)} x^3 dx$$

and which is further [24]

$$\omega_3 \frac{\Gamma^2(2n-2)\Gamma(k+1)\Gamma(k+2)}{\Gamma(k+2n-2)\Gamma(k+2n-1)(2k+2n-1)}.$$

The rest is done by a routine computation. \square

Note that when $\mathbb{F} = \mathbb{R}$ and $n = 3$ our result coincides with that in [22, Lemma 2.4]. For $\mathbb{F} = \mathbb{C}$, and $W^{p,q} = W^{p,0}$ the space the holomorphic polynomials of degree p , the norm of R can be found directly by computing of the integral $\int_S |x_1^p|^2 dx$ on the sphere S in \mathbb{C}^n .

3.4. Discrete components of complementary series. Before stating our first main result we note the following elementary

Lemma 3.5. *Suppose $0 < \alpha < 1$, $\beta > 0$, $\alpha + \beta > 1$ and $\gamma > 1$. Then*

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)^{\alpha}(q+j+1)^{\beta}} \leq C \frac{1}{q^{\alpha+\beta-1}}, \quad \sum_{j=0}^{\infty} \frac{1}{(j+q+1)^{\gamma}} \leq C \frac{1}{(q+1)^{\gamma-1}}, \quad \forall q \geq 0$$

The second estimate is straightforward. The first sum is dominated by the integral

$$\int_0^{\infty} \frac{1}{x^{\alpha}(x+q+1)^{\beta}} dx = \frac{1}{(q+1)^{\alpha+\beta-1}} \int_0^{\infty} \frac{1}{x^{\alpha}(x+1)^{\beta}} dx = \frac{1}{(q+1)^{\alpha+\beta-1}} C$$

since the integral $\int_0^{\infty} \frac{1}{x^{\alpha}(x+1)^{\beta}} dx = C < \infty$ is convergent by our assumption.

Observe also that

$$R : (X_{\nu}, \pi_{\nu}, \mathfrak{g}) \rightarrow (X_{\nu}^{\flat}, \pi_{\nu}^{\flat}, \mathfrak{h}), \quad f(x) \mapsto f(x', 0)$$

intertwines the action of π_{ν}^{\flat} of \mathfrak{h} . Thus the boundedness of R implies that $(\pi_{\nu}^{\flat}, \mathfrak{h})$ is a discrete component whenever both are unitarizable. In accordance with the notation $\|\cdot\|$ in Theorem 3.1 we denote $\|T\|_{\nu, \mu}$ the norm of an operator $T : X_{\nu} \rightarrow X_{\mu}^{\flat}$ and further $\|T\|_{\nu} = \|T\|_{\nu, \nu}$. We have then

$$\|R_{\tau, \sigma}\|_{\nu, \mu}^2 = \frac{\lambda(\sigma)^{\flat}}{\lambda(\tau)} \|R_{\tau, \sigma}\|^2,$$

and the criterion (3.5) becomes

$$(3.10) \quad \sum_{\tau \supset \sigma} \|R_{\tau, \sigma}\|^2 \lambda(\tau)^{-1} \leq \frac{C}{\lambda^{\flat}(\sigma)}.$$

Theorem 3.6. *The restriction of (π_{ν}, G) on H contains (π_{ν}^{\flat}, H) as a discrete component in the following cases*

- (1) $\mathbb{F} = \mathbb{R}$, $n \geq 3$, $0 < \nu < \frac{n-2}{2}$;
- (2) $\mathbb{F} = \mathbb{C}$, $n \geq 3$, $0 < \nu < n-2$;
- (3) $\mathbb{F} = \mathbb{H}$, $n \geq 2$, $2 < \nu < 2n-1$.

Proof. $\mathbb{F} = \mathbb{R}$. First note that $\frac{n-2}{2} < n-2 < n-1$ thus both (π_{ν}, G) and (π_{ν}, H) are well-defined unitary representations. We use now Lemma 3.2. The constants $\lambda(p)$, $\lambda^{\flat}(q)$ and the series (3.10) in questions are

$$\lambda_{\nu}(p) \sim (p+1)^{n-1-2\nu}, \quad \lambda_{\nu}^{\flat}(q) \sim (q+1)^{n-2-2\nu},$$

$$\sum_{p \geq q, p-q \text{ even}} \frac{p+1}{(p+q+1)^{\frac{1}{2}}(p-q+1)^{\frac{1}{2}}} \frac{1}{(p+1)^{n-1-2\nu}}.$$

Writing $p = q + 2j$ we see the sum is dominated by

$$\sum_{j=0}^{\infty} \frac{q+2j+1}{(2q+2j+1)^{\frac{1}{2}}(2j+1)^{\frac{1}{2}}} \frac{1}{(q+2j+1)^{n-1-2\nu}} \leq C \sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{2}}} \frac{1}{(q+j)^{n-1-2\nu-\frac{1}{2}}},$$

and further by $(q+1)^{-(n-2-2\nu)}$ in view of Lemma 3.5, namely by $\frac{1}{\lambda^{\flat}(q)}$.

$\mathbb{F} = \mathbb{C}$. λ_ν has the asymptotics

$$\lambda_\nu(p, q) \sim (p+1)^{n-\nu}(q+1)^{n-\nu}.$$

For fixed type (p_1, q_1) of L the series $\sum_{\tau \supset \sigma} \|R_{\tau, \sigma}\|^2 \lambda(\tau)^{-1}$ is dominated up to a constant by

$$\begin{aligned} & \sum_{p-p_1=q-q_1 \geq 0} \frac{p+q+2}{(p+1)^{n-\nu}(q+1)^{n-\nu}} \\ &= \sum_{p-p_1=q-q_1 \geq 0} \left(\frac{1}{(p+1)^{n-\nu-1}(q+1)^{n-\nu}} + \frac{1}{(p+1)^{n-\nu}(q+1)^{n-\nu-1}} \right) \end{aligned}$$

as sum of two, say $I + II$. Now

$$I = \sum_{k=0}^{\infty} \frac{1}{(p_1+k+1)^{n-\nu-1}(q_1+k+1)^{n-\nu}},$$

and

$$I \leq \frac{1}{(p_1+1)^{n-\nu-1}} \sum_{k=0}^{\infty} \frac{1}{(q_1+k+1)^{n-\nu}} \leq C \frac{1}{(p_1+1)^{n-\nu-1}(q_1+1)^{n-\nu-1}} \leq C \frac{1}{\lambda_\nu^b(p_1, q_1)}$$

by Lemma 3.5. The same holds for II .

$\mathbb{F} = \mathbb{H}$. Writing $p = p_1 + 2k$, $k \geq 0$, we have

$$\lambda_\nu(p, q) \sim (p-q+1)^{2n+1-\nu}(p+q+1)^{2n+1-\nu} \sim (p_1-q+k+1)^{2n+1-\nu}(p_1+q+k+1)^{2n+1-\nu}$$

and

$$\|R_{(p,q),(p_1,q)}\|^2 \sim (k+1)(p_1+k+1)^2.$$

The sum (3.10) is bounded by

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{(k+1)(k+p_1+1)^2}{(p_1-q+k+1)^{2n+1-\nu}(p_1+q+k+1)^{2n+1-\nu}} \\ & \leq \sum_{k=0}^{\infty} \frac{k+1}{(p_1-q+k+1)^{2n+1-\nu}(p_1+q+k+1)^{2n-1-\nu}} \\ & \leq \frac{1}{(p_1+q+1)^{2n-1-\nu}} \sum_{k=0}^{\infty} \frac{k+1}{(p_1-q+k+1)^{2n+1-\nu}} \\ & \leq \frac{1}{(p_1+q+1)^{2n-1-\nu}} \sum_{k=0}^{\infty} \frac{1}{(p_1-q+k+1)^{2n-\nu}} \\ & \leq C \frac{1}{(p_1+q+1)^{2n-1-\nu}} \frac{1}{(p_1-q+k+1)^{2n-\nu-1}} \sim \frac{1}{\lambda_\nu^b(p_1, q_1)}, \end{aligned}$$

finishing the proof. \square

Remark 3.7. For $\mathbb{F} = \mathbb{R}$ and $n = 3$ the full decomposition of the complementary series π_ν of $SO_0(3, 1)$ under $SO_0(2, 1)$ is done in [18]. If (in terms of our parametrization) $\frac{1}{2} \leq \nu < 1$ it is a sum of two direct integrals of spherical principal series, and if $0 < \nu < \frac{1}{2}$ there is one extra discrete component, the complementary series. The appearance of

complementary series π_ν^\flat of $SO_0(n-1, 1)$ in the complementary series π_ν of $SO_0(n, 1)$ is done in [23] using the non-compact realization on \mathbb{R}^{n-1} . It might be possible to find a full decomposition for general $G = SO_0(n, 1; \mathbb{F})$ using the techniques in [18]. The most interesting part might still be the discrete spectrum in view of its stability under restriction and induction in the Ramanujan duals [4, 3].

3.5. The quotients (\mathcal{W}, π_ν) at negative integes ν and their discrete components. The representation π_ν are reducible [12] for ν satisfying certain integral conditions, and there exist unitarizable subrepresentations (or quotients). More precisely we have the following result [25, 12], retaining the notation of λ_τ as the Schur proportional constants; here we have rewritten them in similar formulation as in Theorem 3.1.

Theorem 3.8. *There is a unitarizable irreducible quotient $(\mathcal{W}_\nu, \pi_\nu)$, whose completion forms a unitary irreducible representation of G in the following cases*

- (1) $\mathbb{F} = \mathbb{R}$, $n \geq 3$, $\nu = -k$, $k \geq 0$, $\mathcal{W}_\nu = X_\nu/M_\nu$,

$$M_\nu = \sum_{p=0}^k W^p,$$

$$\lambda_\nu(p) = \frac{(n-1-\nu+k+1)_{p-k-1}}{(\nu+k+1)_{p-k-1}} = C_\nu \frac{\Gamma(n-1-\nu+p)}{\Gamma(\nu+p)};$$

- (2) $\mathbb{F} = \mathbb{C}$, $n \geq 2$, $\nu = -2k$, $k > 0$, $\mathcal{W}_\nu = X_\nu/M_\nu$,

$$M_\nu = \sum_{p \leq k, q \geq 0} W^{p,q} + \sum_{q \leq k, p \geq 0} W^{p,q} \quad (k > 0),$$

$$\begin{aligned} \lambda_\nu(p, q) &= \frac{(n - \frac{\nu}{2} + k + 1)_{p-k-1} (n - \frac{\nu}{2} + k + 1)_{q-k-1}}{(\frac{\nu}{2} + k + 1)_{p-k-1} (\frac{\nu}{2} + k + 1)_{q-k-1}} \\ &= C_\nu \frac{\Gamma(n - \frac{\nu}{2} + p) \Gamma(n - \frac{\nu}{2} + q)}{\Gamma(\frac{\nu}{2} + p) \Gamma(\frac{\nu}{2} + q)} \end{aligned}$$

and for $k = 0$ with three quotients $(\mathcal{W}_0^\pm, \pi_0^\pm)$, (\mathcal{W}_0, π_0) ,

$$\mathcal{W}_0^+ = \sum_{p=0}^{\infty} W^{p,0}/\mathbb{C}, \quad \mathcal{W}_0^- = \sum_{q=0}^{\infty} W^{0,q}/\mathbb{C}$$

$$\lambda_\nu^+(p) = \frac{\Gamma(p)}{\Gamma(n+p)}, \quad \lambda_\nu^-(q) = \frac{\Gamma(q)}{\Gamma(n+q)},$$

and

$$\mathcal{W}_0 = X_0 / \sum_{p=0}^{\infty} (W^{p,0} + W^{0,p})$$

$$\lambda_0(p, q) = \frac{\Gamma(n+p-1) \Gamma(n+q-1)}{\Gamma(p) \Gamma(q)};$$

(3) $\mathbb{F} = \mathbb{H}$, $n \geq 1$, $\nu = \nu = -2k$, $k \geq -1$, $\mathcal{W}_\nu = X_\nu/M_\nu$,

$$\begin{aligned} M_\nu &= \sum_{p-q \leq 2k+2} W^{p,q}, \quad k \geq 0, & M_\nu &= \sum_{p=q=0}^{\infty} W^{p,q}, \quad k = 0, \\ \lambda_\nu(p, q) &= \frac{(2n - \frac{\nu}{2} + k + 1)_{\frac{p-q}{2}-k-1} (2n + 1 - \frac{\nu}{2} + k + 1)_{\frac{p+q}{2}-k-1}}{(\frac{\nu}{2} - 1 + k + 1)_{\frac{p-q}{2}-k-1} (\frac{\nu}{2} + k + 1)_{\frac{p+q}{2}-k-1}} \\ &= C_\nu \frac{\Gamma(2n - \frac{\nu}{2} + \frac{p-q}{2}) \Gamma(2n + 1 - \frac{\nu}{2} + \frac{p+q}{2})}{\Gamma(\frac{\nu}{2} - 1 + \frac{p-q}{2}) \Gamma(\frac{\nu}{2} + \frac{p+q}{2})}. \end{aligned}$$

Note that the same ν as above is also a reducible point for $(X_{\nu, \flat}, \pi_\nu^\flat, \mathfrak{h})$. The corresponding quotient representation for \mathfrak{h} will be written as $(\mathcal{V}_k, \pi_\nu^\flat, \mathfrak{h})$

Theorem 3.9. *Let $n \geq 4$ for $\mathbb{F} = \mathbb{R}$, $n \geq 3$ for $\mathbb{F} = \mathbb{C}$, and $n \geq 2$ for $\mathbb{F} = \mathbb{H}$. The representation $(\mathcal{V}_k, \pi_\nu^\flat, \mathfrak{h})$ (and the corresponding completion as for H) appears as an irreducible discrete component in $(\mathcal{W}_k, \pi_{\nu(k)}, \mathfrak{g})$ (respectively for G) restricted to \mathfrak{h} (resp. H).*

Proof. Let $Q = Q_\nu$ be the quotient map $Q : X_\nu^\flat \rightarrow X_\nu^\flat/M_\nu^\flat := \mathcal{V}_k$ at the reducible point ν as above for the group H . The map $QR : X_\nu \rightarrow X_\nu^\flat \rightarrow \mathcal{V}_k$ is clearly $(\pi_\nu^\flat, \mathfrak{h})$ intertwining and induces a map

$$QR : \mathcal{W}^k = X_\nu/M_\nu \rightarrow X_\nu^\flat/M_\nu^\flat = \mathcal{V}_k.$$

We prove the boundedness of QR by the method above. Notice that the asymptotics exponent of $\lambda(\nu)$ has the same dependence for positive ν , e.g. in the case $\mathbb{F} = \mathbb{R}$ with $\nu = -k$,

$$\lambda_\nu(p) \sim (p+1)^{n-1-2\nu}, \quad p \geq k+1,$$

and $n-1-2\nu \geq 2$. Thus the same proof carries over to all cases, and we omit the details. \square

There is some slightly difference when $n = 3$ for $\mathbb{F} = \mathbb{R}$, as $H = SO_0(2, 1)$ has its maximal compact subgroup being the torus and there is a splitting of the restriction to holomorphic and antiholomorphic discrete series. Note that we have also excluded the case $\mathbb{F} = \mathbb{C}$, $n = 2$, namely $SU(2, 1)$, as the restriction map above is zero on the quotient \mathcal{W}_{-2k} ; actually \mathcal{W}_{-2k} is a discrete series and its branching under $SU(1, 1)$ can possibly be studied using some general tools [19, 15].

3.6. The representation π_0 and π_0^\pm for $SU(n, 1)$. The representation π^\pm on the quotient

$$\pi_0^+ = \sum_{p=0}^{\infty} W^{p,0}/\mathbb{C}, \quad \pi_0^- = \sum_{p=0}^{\infty} W^{0,p}/\mathbb{C}$$

are unitarizable representations of \mathfrak{g} . π_0^+ can be constructed also by using the analytic continuation of the weighted Bergman space [7, Theorem 5.4], i.e. scalar holomorphic discrete series, on the unit ball G/K in \mathbb{C}^n with reproducing kernel $(1 - (z, w))^{-\mu}$ at

the reducible point $\mu = 0$; see e.g. [10] where a reproducing kernel and its expansion are found for the space. A full decomposition under $SU(n-1, 1)$ of the series and their quotient can be obtained easily. Indeed let $\pi_0^{\pm, b}$ be the corresponding representation for H and $\pi_j^{\pm, b}$ the unitary representation of H realized as the space of holomorphic functions on the unit ball $\{z \in \mathbb{C}^{n-1}, |z| < 1\}$ with reproducing kernel $(1 - (z, w))^j$, with H acting as

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in H, f(z) \mapsto (cz + d)^{-j} f((az + b)(cz + d)^{-1}).$$

$\pi_{j, b}^+$ is a discrete series of H only when $j \geq n$. Define analogously $\pi_j^{-, b}$ in terms of conjugate holomorphic functions. The following result follows from easy consideration of expansion of holomorphic functions $f(x)$ in the last variable x_n .

Proposition 3.10. *The representation (π_0^{\pm}, G) is decomposed under H as*

$$\pi_0^{\pm} = \pi_0^{\pm, b} \oplus \left(\sum_{j=1}^{\infty} \pi_j^{\pm, b} \right).$$

4. RESTRICTION OF $(F_{4(-20)}, \pi_{\nu})$ TO $H = Spin(8, 1)$

4.1. The subgroup $Spin(8, 1)$. Recall that $H_0 \in \mathfrak{p} = \mathbb{O}^2$ has nonzero roots $\pm 2, \pm 1$ in \mathfrak{g} . Denote $\mathfrak{g}_{\pm 2}$ and $\mathfrak{g}_{\pm 1}$ the respective root spaces. Then the Lie algebras \mathfrak{g}_{\pm} generate a subalgebra of \mathfrak{g} of rank one which is easily seen to be $\mathfrak{h} := spin(8, 1)$. The Cartan decomposition of \mathfrak{h} is $\mathfrak{h} = spin(8) \oplus \mathbb{O}$ with $spin(8)$ acting on \mathbb{O} by the spin representation. The simply connected subgroup of G with Lie algebra \mathfrak{h} is then $H = Spin(8, 1)$ whose maximal compact group is $L = Spin(8)$; see e.g. [3]. It follows from the decomposition (2.5) that the stabilizer of $H_0 \in \mathbb{O} \subset \mathfrak{h}$ in H is also $M = Spin(7)$ and that $L/M = Spin(8)/Spin(7)$ is the sphere S^7 . We have thus

$$L^2(S^7) = \sum_{p \geq 0}^{\oplus} V^p$$

where V^p is the space of spherical harmonics of degree p on S^7 , defined by the condition $x_2 = 0$ in $S^{15} = \{x = (x_1, x_2) \in \mathbb{O}^2; |x| = 1\}$. The decomposition of $L^2(S^{15}) = L^2(K/M)$ is given in (2.6) with $(W^{p, q})^M = \mathbb{C}\phi_{p, q}$. We consider now the restriction of $W^{p, q}|_{x_2=0}$ of the components $W^{p, q}$.

Lemma 4.1. *The decomposition of $W^{p, q}$ under $Spin(8)$ is multiplicity free and $W^{p, q}|_{x_2=0} = V^q$; in other words the only irreducible component in the decomposition with non-zero restriction to S^7 is the representation V^q . Moreover the square norm of $R : W^{p, q} \rightarrow V^q$ is given by*

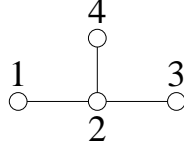
$$\begin{aligned} \|R\|^2 &= C \frac{(p+7) \prod_{j=0}^2 (p+q+8+2j)(p-q+2+2j)(q+4+2j)(q+1+2j)}{(q+3)(q+1)_5} \\ &\sim (p+1)(p+q+1)^3(p-q+1)^3 \end{aligned}$$

where C is a numerical constant independent of p and q .

Remark 4.2. The representation $W^{p,q}$ of $K = Spin(9)$ is of highest weight $\frac{p-q}{2}\lambda_1 + q\lambda_4$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ the fundamental weights dual to the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. In the standard notation they are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4, \alpha_4 = e_4$ and $\frac{p-q}{2}\lambda_1 + q\lambda_4 = \frac{p}{2}e_1 + \frac{q}{2}e_2 + \frac{q}{2}e_3 + \frac{q}{2}e_4 = \frac{1}{2}(p, q, q, q)$. The four simple roots for $spin(8)$ are $\delta_1 = e_1 - e_2, \delta_2 = e_2 - e_3, \delta_3 = e_3 - e_4, \delta_4 = e_3 + e_4$. The above branching rule above can be formulated as

$$W^{\frac{1}{2}(p,q,q,q)}|_{x_2=0} = V^{\frac{q}{2}(1,1,1,1)},$$

with $V^{\frac{q}{2}(1,1,1,1)}$ being the space of spherical harmonics of degree q on S^7 , which is also of highest weight qe_1 as $SO(8)$ representation. This discrepancy of highest weights is explained by the triality in $Spin(8)$. The Dynkin diagram of $Spin(8)$ is



There is a symmetry of S_3 (as outer automorphisms) acting on the three simple roots $\delta_1, \delta_3, \delta_4$. The highest weight $\frac{q}{2}(1, 1, 1, 1) = \frac{q}{2}(\delta_1 + 2\delta_2 + \delta_3 + 2\delta_4)$, whereas $qe_1 = \frac{q}{2}(2\delta_1 + 2\delta_2 + \delta_3 + \delta_4)$ and the permutation (134) exchanges the two weights. Also the multiplicity one property of $W^{p,q}$ under $M = Spin(7)$ factors through $Spin(8)$ and we have $(W^{p,q})^M|_{x_2=0} = (W^{p,q}|_{x_2=0})^M = (V^q)^M = \phi_q^8$. Note that $M = Spin(7)$ in $Spin(8) \subset Spin(9)$ is not the obvious copy of $Spin(7)$ in $Spin(9)$ defined by the standard inclusion $\mathbb{R}^7 \subset \mathbb{R}^8 \subset \mathbb{R}^9$; in the space $W^{p,q}$ the former copy $M = Spin(7)$ has only one-dimensional fixed vectors, whereas the latter copy $Spin(7)$ has arbitrarily large multiplicities by the construction of Gelfand-Zetlin basis [27].

Proof. The first statement is well-known. Any irreducible representation of $Spin(8)$ in $W^{p,q}|_{x_2=0}$ is a constituent in $L^2(S^7)$ and contains thus a unique $M = Spin(7)$ -invariant element. But $(W^{p,q}|_{x_2=0})^M = (W^{p,q})^M|_{x_2=0} = \mathbb{C}\phi_{p,q}^8|_{x_2=0}$, and $\phi_{p,q}^8|_{x_2=0}$ is

$$\phi_{p,q}(\cos \eta, 0) = \cos^q \eta F\left(-\frac{q}{2}, -\frac{q-1}{2}; \frac{7}{2}; -\tan^2 \eta\right)$$

which is precisely the M -invariant spherical harmonics $\phi_q^8(\cos \eta)$ on S^7 , Section 2.2. Thus $W^{p,q}|_{x_2=0}$ is nonzero and is just V^q . In particular the element $\phi_{p,q}$ is in the isotypic component $\tilde{V}^q \subset W^{p,q}$ of V^q . The squared norm of R on $W^{p,q}$ is

$$\|R\|^2 = \|R\phi_{p,q}\|^2 \phi_{p,q}^{-2}, \quad R\phi_{p,q} = \phi_q^8.$$

Both norms can be evaluated by the dimension formula. Following the notation in the above remark we have $W^{p,q}$ has highest weight $\frac{p}{2}e_1 + \frac{q}{2}e_2 + \frac{q}{2}e_3 + \frac{q}{2}e_4$ with the positive roots being $\{e_i \pm e_j, e_i, 1 \leq i < j \leq 4\}$, and the dimension of $W^{p,q}$ is then

$$\begin{aligned} \dim W^{p,q} &= C_1(p+7) \prod_{j=0}^2 (p+q+8+2j)(p-q+2+2j)(q+4+2j)(q+1+2j) \\ &\sim (p+1)(p+q+1)^3(p-q+1)^3(q+1)^6 \end{aligned}$$

whereas the dimension of V^q is

$$\dim V^q = C_2(q+3)(q+1)_5 \sim (q+1)^6$$

for some constants C_1, C_2 independent of p and q . This completes the proof. \square

4.2. Discrete components. Define the principal series representation π_ν of G as in (3.1), realized on $L^2(K/M) = L^2(S^{15})$. We recall results in [11] on the spherical complementary series of G .

Theorem 4.3. *Let $6 < \nu < 16$. There is a positive definite (\mathfrak{g}, π_ν) -invariant form on the (\mathfrak{g}, K) -module $\sum_{p,q} W^{p,q}$ defined by*

$$\|w\|_\nu = \sqrt{\lambda_\nu(p, q)} \|w\|, \quad \lambda_\nu(p, q) = \frac{(8 - \frac{\nu}{2})_{\frac{p-q}{2}} (11 - \frac{\nu}{2})_{\frac{p+q}{2}}}{(\frac{\nu}{2} - 3)_{\frac{p-q}{2}} (\frac{\nu}{2})_{\frac{p+q}{2}}}.$$

We study now the branching of the complementary series under $H = Spin(8, 1)$. Denote π_μ^b the principal series representation of $SO_0(8, 1)$, thus also for H , as defined for in (3.1). Note that there is a discrepancy between the normalization of H_0 here and there; the roots of H_0 in \mathfrak{h} is ± 2 here instead of ± 1 as in Section 2.1. In particular, the restriction map $R : f(x_1, x_2) \mapsto Rf(x_1) = f(x_1, 0)$, $(x_1, x_2) \in \mathbb{O}^2$, defines an \mathfrak{h} -intertwining operator

$$R : (X_\nu, \pi_\nu, \mathfrak{g}) \rightarrow (X_{\frac{\nu}{2}}^b, \pi_{\frac{\nu}{2}}^b, \mathfrak{h}).$$

Theorem 4.4. *Let $6 < \nu < 7$. The restriction of (π_ν, G) on H contains $(\pi_{\frac{\nu}{2}}^b, H)$ as a discrete component.*

Proof. The λ_ν and λ_μ^b in this case are

$$\lambda_\nu(p, q) \sim (p-q+1)^{11-\nu} (p+q+1)^{11-\nu}, \quad \lambda_\mu(q) \sim (q+1)^{7-2\mu}$$

with $p - q = 2k \geq 0$ even. The sum to be treated is

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{q+k+1}{(k+1)^{8-\nu}(2k+q+1)^{8-\nu}} \\
& \leq \sum_{k=0}^{\infty} \frac{1}{(k+1)^{8-\nu}(2k+q+1)^{7-\nu}} \\
& \leq \frac{1}{(q+1)^{7-\nu}} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{8-\nu}} \\
& = C \frac{1}{(q+1)^{7-\nu}} \sim \frac{1}{\lambda_{\frac{\nu}{2}}^{\nu}(q)},
\end{aligned}$$

completing the proof. \square

Remark 4.5. The complementary series in is parametrized in [3, Example C] as $-5 \leq \lambda \leq 5$, i.e. the standard parametrization [14]. Our ν is their $\rho + \lambda = 11 + \lambda$. It is stated there that the point $\lambda = 3$, i.e. $\nu = 8$ is in the automorphic dual \hat{G}_{aut} of G . Note that this point falls outside the range $6 < \nu < 7$ in our theorem. One can deduce from the Burger-Li-Sarnak conjecture on the Ramanujan dual \hat{H}_{Raman} for $H = SO(n, 1)$ and our theorem above on some nonexistence of certain intervals in the set \hat{H}_{Raman} . In view of [3, Theorem 1] it would be also interesting to study the induction of automorphic representations of H to G .

The representation π_{ν} has also unitarizable subquotients at integral ν : for $\nu = 6 - 2k$, $k \geq 0$, the quotient

$$\mathcal{W}^{\nu} = X_{\nu}/M_{\nu}, \quad M_{\nu} = \sum_{p-q \leq k} W^{p,q},$$

is unitarizable; see [11]. However in this case the restriction composed with quotient map is zero. Presumably there is no discrete component under H and it would be interesting to pursue this further.

The main results in the present paper prove the existence of one single discrete component under H of a complementary series of G . It may happen that there are more discrete components. We shall study them in a forthcoming paper. Applying the stability result [3, Theorem 1 (ii)] to those cases where π_{ν}^b appear discretely in π_{ν} we conclude finally if π_{ν} is an automorphic representation of G so is π_{ν}^b .

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